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Asymptotic properties of two interacting maps

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Abstract. In this paper we consider a system whose state x changes to $\sigma(x)$ if a perturbation occurs at the time t, for $t > 0, t \notin \mathbb{N}$. Moreover, the state x changes to the new state $\eta(x)$ at time t, for $t \in \mathbb{N}$. It is assumed that the number of perturbations in an interval $(0, t)$ is a Poisson process. Here η and σ are measurable maps from a measure space $(E, \mathcal{A}, \mu)$ into itself. We give conditions for the existence of a stationary distribution of the system when the maps η and σ commute, and we prove that any stationary distribution is an invariant measure of these maps.

Keywords: Alternating maps, stationary distribution, discrete dynamical system.

Mathematical subject classification: 34F05, 37C30.

1 Introduction

The issue of random perturbations in dynamical systems has received considerable attention. This is an interesting question both from the theoretical point of view and for the applications. An important problem in this matter arises when considering a deterministic dynamical system where some of its parameters varies in an unpredictable way. Mathematical ecology is a rich source of examples of this kind of problems, where it is frequent to assume that the size of a population is determined by a discrete dynamical system of the form $x_n = F(x_{n-1}; r)$. The parameter r changes randomly in a set R, due to, say, environmental fluctuations. The simplest case is when the set R has only two elements.

Many interesting problems have been studied in this area. Some of them are concerned with the balance between chaos and randomness. More precisely, can the random choice of the parameter r damp the chaotic fluctuations of one of the maps? About this topic, see for example [DA], [HT], [LM], [P], [SEK], [St]. In this paper we consider a system whose state x changes to $\sigma(x)$ if a perturbation occurs at the time t, for $t > 0, t \notin \mathbb{N}$. Moreover, the state x changes to the new

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state $\eta(x)$ at integer time $t \in \mathbb{N}$. It is assumed that the number of perturbations in an interval $(0, t)$ is a Poisson process. Here η and σ are measurable maps from a measure space $(E, \mathcal{A}, \mu)$ into itself. We are interested in the relation between the stationary distribution of the system and the corresponding ones for the maps η and σ.

In Section 2 we give some basic definitions and we describe a heuristical approach to get a differential equation in $L^1(E, \mathcal{A}, \mu)$ whose solution $T_t(f)$ is the density of the probability distribution of the state of the system at time t, when f is the initial density.

The main results are stated in Section 4. Here, we prove that any stationary distribution of the system is always a common stationary distribution of both η and σ (see Theorems 4.1 and 4.2). The frequency of perturbation only contributes to the choice of this common distribution. It is important to work in a L^1-space as we see from Example 4.2, where we consider two commutative affine maps. In this case, it is possible to have a stationary distribution for certain frequencies of the perturbations, not withstanding the fact that one of the maps has no stationary distribution.

These results show an interesting interaction between the dynamics of two commuting maps. Roughly speaking, the asymptotic evolution of a discrete dynamical system cannot be affected by perturbations produced by a commutative map. The likelihood of two commuting maps is also investigated in [S], through the analysis of the orbits for each map.

2 The Mathematical Model

We consider a discrete dynamical system given by

$$x_{k+1} = \eta(x_k), \qquad k \in \mathbb{N},$$

where $\eta : E \to E$ is a continuous map defined in a measure space $(E, \mathcal{A}, \mu)$. We assume that in every interval $(k, k+1)$ perturbations occur according to a Poisson distribution; that is, the probability to have l fluctuations in an interval $(0, t)$ is

$$P_l(t) = \frac{(\lambda t)^l}{l!} e^{-\lambda t}. \tag{1}$$

After a fluctuation, the state x of the system changes to $\sigma(x)$. Thus, if the state of the system at time k is $\eta(x_k)$, and a perturbation occurs at time $k+t$, with $0 < t < 1$, then the new state is $\sigma(\eta(x_k))$. Here, $\sigma : E \to E$ is a non-singular map. That means that $\mu(A) = 0$ implies $\mu(\sigma^{-1}(A)) = 0$.

In this way, we obtain a stochastic process $\{X_t\}_{t\in\mathbb{R}^+}$, where X_t is the state of the system at time t. The probability distribution of this process is $P(X_t \in A) = \int_A u(t,x)d\mu(x)$, where $u(t,x)$ is the density of the probability distribution at time t, if it exists.

Let $f : E \to \mathbb{R}^+$ be the density of an initial probability distribution of the states of the system and A a measurable subset of E. The increment of the probability that the state of the system belongs to A during an interval Δt at time t is given by

$$P(X_{t+\Delta t} \in A) - P(X_t \in A) = \int_A u(t+\Delta t, x)\, d\mu(x) - \int_A u(t,x)d\mu(x), \tag{2}$$

where t and $t+\Delta t$ are in $[k, k+1)$.

We take Δt small enough, so that the probability to have more than one perturbation is negligible. The probability that the state of the system belongs to A after a fluctuation is

$$\lambda\Delta t \int_{\sigma^{-1}(A)} u(t,x)\, d\mu(x), \tag{3}$$

since $P_1(\Delta t) = \lambda\Delta t e^{-\lambda\Delta t} \approx \lambda\Delta t$. Therefore, the increment of the probability that a state remains in A after a perturbation is given by

$$\lambda\Delta t \int_{\sigma^{-1}(A)} u(t,x)\, d\mu(x) - \lambda\Delta t \int_A u(t,x)d\mu(x). \tag{4}$$

From equations (2) and (4) we get the following relation

$$\int_A [u(t+\Delta t, x) - u(t,x)]\, d\mu(x) = \lambda\Delta t \left(\int_{\sigma^{-1}(A)} u(t,x)d\mu(x) - \int_A u(t,x)d\mu(x) \right).$$

The above equation can be written as

$$\int_A [u(t+\Delta t, x) - u(t,x)]\, d\mu(x) = \lambda\Delta t \int_A [-u(t,x) + P_\sigma\, u(t,x)]d\mu(x),$$

where P_σ is the Frobenius–Perron operator corresponding to the map σ. Dividing the above equation by Δt and passing to the limit $\Delta t \to 0$, we obtain

$$\int_A \frac{\partial u(t,x)}{\partial t} d\mu(x) = \lambda \int_A [-u(t,x) + P_\sigma u(t,x)]d\mu(x). \tag{5}$$

From this, we have

$$\frac{\partial u(t,x)}{\partial t} = -\lambda\,[u(t,x) - P_\sigma u(t,x)]$$

for almost all $x \in E$, and for all $t \in [k, k+1)$.

In order to obtain the density of the probability distribution $u(t,x)$ we have to solve the following initial condition problem

$$\frac{\partial u(t,x)}{\partial t} = -\lambda\,[u(t,x) - P_\sigma u(t,x)]\,, \tag{6}$$

with initial condition $u(0,x) = f(x)$, for $t \in [0,1)$ and $u(i,x) = P_\eta(u^-(i,x)))$, for $t \in [i, i+1)$, $i \in \mathbb{N}$. Here $u^-(i,x)$ denotes the left hand side limit of $u(t,x)$ when t approaches i.

3 Solution of the Model

After a change of scale, equation (6) can be written in the form

$$\frac{\partial u(t,x)}{\partial t} = (P_\sigma - I)u(t,x), \tag{7}$$

and the initial conditions take the form

$$u(i\lambda, x) = \begin{cases} f(x), & t \in [0,\lambda), \\ P_\eta(u^-(i\lambda, x)) & \text{for t} \in [i\lambda, (i+1)\lambda), \end{cases} \tag{8}$$

where $i \in \mathbb{N} \cup \{0\}$.

It is well known [DS] that the solution of equation (7) with initial condition $u(0,x) = f(x)$ is a semigroup of operators $\{S_t\}_{t\geq 0}$ in the space $L^1(\mu)$ given by

$$S_t f = e^{t(P_\sigma - I)}\, f = e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}P_\sigma^n f.$$

Then it follows easily that the solution T_t of equation (7) with initial conditions (8) coincides with S_t in the interval $[0,\lambda)$. Notice that $u^-(\lambda, x)$ satisfies

$$u^-(\lambda,\cdot) = \lim_{t\to\lambda^-} S_t f,$$

so, $T_\lambda f(x) = P_\eta(u^-(\lambda, x))$. Thus, if we make the change of variable $\tau = t - \lambda$ in (7) and (8) we obtain

$$T_t f = e^{(t-\lambda)(P_\sigma - I)}\,(P_\eta(e^{(P_\sigma - I)}\, f)) = S_\tau(P_\eta(S_\lambda f)),$$

for $t \in [\lambda, 2\lambda)$. In general, the solution is

$$T_t f = S_\tau(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f)}^{k-times}), \qquad \tau = t - k\lambda,$$

for $t \in [k\lambda, (k+1)\lambda)$, $k \in \mathbb{N} \cup \{0\}$.

4 Stationary distributions of the Model

Lemma 4.1. *For any* $f_1, f_2 \in L^1$ *we have that*

$$|| T_t f_1 - T_t f_2 || \leq || f_1 - f_2 ||$$

for all time $t \geq 0$.

Proof. We consider $t \in [(m-1)\lambda, m\lambda)$ with $m \in \mathbb{N}$ and we do induction over m.

Let f_1, f_2 be functions in L^1. If $m = 1$ then,

$$||T_t f_1 - T_t f_2|| = ||S_t f_1 - S_t f_2|| \leq ||S_t|| ||f_1 - f_2|| \leq ||f_1 - f_2||.$$

So the inequality holds for $t \in [0, \lambda)$.

Now, we assume that the inequality holds for $m \leq k$ and we prove that it is true for $m = k+1$. In this case

$$\begin{aligned} ||T_t f_1 - T_t f_2|| &= ||S_\tau(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f_1)}^{k-times}) - S_\tau(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f_2)}^{k-times})|| \\ &\leq || \overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f_1)}^{k-times} - \overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f_2)}^{k-times} || \\ &\leq || \overbrace{S_\lambda P_\eta \cdots S_\lambda(f_1)}^{k-times} - \overbrace{S_\lambda P_\eta \cdots S_\lambda(f_2)}^{k-times} || \end{aligned}$$

The last two inequalities follow from the properties of the semigroup S_t and the operator P_η, respectively.

Finally, by the induction hypothesis we have,

$$||T_t f_1 - T_t f_2|| \leq ||f_1 - f_2||, \quad \text{for } t \in [0, (k+1)\lambda).$$

So for every $t \geq 0$ and for all $f_1, f_2 \in L^1$

$$||T_t f_1 - T_t f_2|| \leq ||f_1 - f_2||. \qquad \square$$

Proposition 4.1. *If η and σ commute, then the operators P_η and P_σ also commute.*

Proof. Let $A \subset E$ be a measurable set. Then

$$\begin{aligned}
\int_A P_\sigma(P_\eta f) d\mu(x) &= \int_{\sigma^{-1}(A)} P_\eta f d\mu(x) = \int_{\eta^{-1}(\sigma^{-1}(A))} f d\mu(x) \\
&= \int_{(\sigma\circ\eta)^{-1}(A)} f d\mu(x) = \int_{(\eta\circ\sigma)^{-1}(A)} f d\mu(x) \\
&= \int_{\sigma^{-1}(\eta^{-1}(A))} f d\mu(x) = \int_{\eta^{-1}(A)} P_\sigma f d\mu(x) \\
&= \int_A P_\eta P_\sigma f d\mu(x).
\end{aligned}$$

Since this is true for any $f \in L^1(E)$ and for every measurable set, we have that $P_\sigma P_\eta = P_\eta P_\sigma$. □

It is clear that the reciprocal is false.

Lemma 4.2. *If the operators P_η and P_σ commute, then*

$$T_t f = S_t(P_\eta^k f) \quad \textit{for all } t \in [k\lambda, (k+1)\lambda).$$

Proof. If $t \in [0, \lambda)$, then

$$T_t f = S_t(f) = S_t(P_\eta^0 f).$$

So the statement holds for $k = 0$.

If $k = 1$, then $t \in [\lambda, 2\lambda)$ and we have

$$\begin{aligned}
T_t f &= S_{t-\lambda}(P_\eta(S_\lambda f)) = S_{t-\lambda} P_\eta (e^{-\lambda} \textstyle\sum_{n=0}^{\infty} \dfrac{\lambda^n}{n!} P_\sigma^n f) \\
&= S_{t-\lambda}(e^{-\lambda} \textstyle\sum_{n=0}^{\infty} \dfrac{\lambda^n}{n!} P_\sigma^n (P_\eta f) \\
&= S_{t-\lambda}(S_\lambda(P_\eta f)) = S_t(P_\eta f).
\end{aligned}$$

Now, suppose that the statement holds for $k \leq m$, *i. e.* that

$$T_t f = S_t(P_\eta^k f) \quad \forall\, t \in [k\lambda, (k+1)\lambda),$$

for every $k = 0, 1, \ldots, m$. We will demonstrate that this equality is true for $k = m + 1$. If t belongs to $[(m+1)\lambda,\ (m+2)\lambda)$, then

$$\begin{aligned}
T_t f &= S_{t-(m+1)\lambda}(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda}^{(m+1)-times}) f \\
&= S_{t-(m+1)\lambda}(P_\eta S_\lambda) T_{m\lambda} f \\
&= S_{t-(m+1)\lambda}(P_\eta S_\lambda) S_{m\lambda} P_\eta^m f \\
&= S_{t-(m+1)\lambda} S_{(m+1)\lambda} P_\eta^{m+1} f \\
&= S_t P_\eta^{m+1} f.
\end{aligned}$$

Thus, the statement holds for all $k \in \mathbb{N}$ and we have that

$$T_t f = S_t(P_\eta^k f) \quad \text{for} \quad t \in [k\lambda, (k+1)\lambda) \qquad \square$$

Lemma 4.3.

(a) *Given $f \in L^1$ the norm*

$$||T_t f||$$

is a non increasing function of time t.

(b) *If P_η and P_σ commute then*

$$||T_{t+m} f|| \leq ||T_t(T_m f)||,$$

for $t, m \in \mathbb{R}^+ \cup \{0\}$.

Proof. To prove this lemma we use that for all $f_1, f_2 \in L^1$

$$||S_t f_1 - S_t f_2||$$

is a non increasing function of time t and $||P_\eta f_1|| \leq ||f_1||$ (see [LM]).

(a) Suppose that $t_1 < t_2$, $t_1 = k_1\lambda + \tau_1$ and $t_2 = k_2\lambda + \tau_2$ where $k_1, k_2 \in \mathbb{N}$ and $\tau_1, \tau_2 \in [0, \lambda)$.

In the case that $k_1 = k_2$ then $\tau_1 < \tau_2$ and

$$\begin{aligned}
||T_{t_2} f|| &= ||S_{\tau_2}(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f)}^{k_2-times})|| \\
&\leq ||S_{\tau_1}(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f)}^{k_1-times})|| \\
&= \|T_{t_1} f\|
\end{aligned}$$

If $k_1 \neq k_2$ then there exists $j \in \mathbb{N}$ such that $k_2 = k_1 + j$ and

$$\begin{aligned}
||T_{t_2} f|| &= ||S_{\tau_2}(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f)}^{(k_1+j)-times})|| \\
&\leq ||S_\lambda(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f)}^{k_1-times})|| \\
&\leq ||S_{\tau_1}(\overbrace{P_\eta S_\lambda \cdots P_\eta S_\lambda(f)}^{k_1-times})|| \\
&= \|T_{t_1} f\|.
\end{aligned}$$

(b) We assume that $t = k_1\lambda + \tau_1$, $m = k_2\lambda + \tau_2$, where $k_1, k_2 \in \mathbb{N}$ and $\tau_1, \tau_2 \in [0, \lambda)$.

Since P_η commutes with P_σ

$$||T_{t+m} f|| = ||S_{t+m}(P_\eta^k f)||,$$

where $t + m = (k_1 + k_2)\lambda + \tau_1 + \tau_2$, $0 \leq \tau_1 + \tau_2 < 2\lambda$ and $k = k_1 + k_2 + r$ for some $r \in \{0, 1\}$. Hence,

$$\begin{aligned} ||T_{t+m} f|| &= \|S_{t+m}(P_\eta^{k_1+k_2+r} f)\| \\ &= ||P_\eta^r S_{t+m}(P_\eta^{k_1+k_2} f)|| \\ &\leq ||S_{t+m}(P_\eta^{k_1+k_2} f)|| \\ &= ||S_t(S_m P_\eta^{k_1+k_2} f)|| \\ &= ||S_t P_\eta^{k_1}(S_m P_\eta^{k_2} f)|| \\ &= ||T_t(T_m f)|| \end{aligned}$$

□

Lemma 4.4. *Suppose that P_η and P_σ are commutative. Then,*

$$\lim_{t\to\infty} [T_t(P_\sigma f) - T_t f] = 0$$

for each $f \in L^1$.

Proof. To prove this we closely follow [LM, Section 8.5]. As P_η and P_σ commute, by lemma 4.2 we have

$$\begin{aligned} T_t f &= S_t(P_\eta^k f) \quad \text{when} \quad t \in [k\lambda, (k+1)\lambda) \\ &= e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} P_\sigma^n (P_\eta^k f) \end{aligned}$$

and

$$\begin{aligned} T_t(P_\sigma f) &= S_t(P_\sigma P_\eta^k f) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} P_\sigma^{n+1}(P_\eta^k f) \\ &= e^{-t} \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} P_\sigma^n \left(P_\eta^k f\right). \end{aligned}$$

Therefore,

$$\begin{aligned}\|T_t(P_\sigma f) - T_t f\| &= \|e^{-t}\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}P_\sigma^n(P_\eta^k f) - e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}P_\sigma^n(P_\eta^k f)\| \\ &= e^{-t}\|P_\eta^k(\sum_{n=0}^{\infty}\frac{t^n}{n!}P_\sigma^n f - \sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}P_\sigma^n f)\| \\ &\le e^{-t}\|\sum_{n=1}^{\infty}(\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!})P_\sigma^n f + f\|.\end{aligned}$$

If $t = m$ and $m \in \mathbb{N}$, then

$$e^{-t}\sum_{n=1}^{\infty}\left|\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!}\right| = 2e^{-m}(\frac{m^m}{m!} - \frac{1}{2})$$

converges to zero when m goes to infinity. Therefore

$$e^{-t}\sum_{n=1}^{\infty}\left|\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!}\right|$$

converges to zero when t goes to infinity for all $t \in \mathbb{N}$.

As $||T_t f - T_t(P_\sigma f)||$ is a non increasing function of time t (see lemma 4.3), we have the convergence for all $t \ge 0$. □

Lemma 4.5. *If P_σ commutes with P_η, then the operators P_η, P_σ and T_t commute.*

Proof. Let f be a function in L^1. From the continuity of P_σ we have that

$$P_\sigma(S_t f) = S_t(P_\sigma f).$$

By lemma 4.2,

$$T_t f = S_t(P_\eta^k f) \quad t \in [k\lambda, (k+1)\lambda).$$

Therefore

$$\begin{aligned}P_\sigma(T_t f) &= P_\sigma(S_t(P_\eta^k f)) = S_t(P_\sigma(P_\eta^k f)) \\ &= S_t(P_\eta^k(P_\sigma f)) = T_t(P_\sigma f).\end{aligned}$$ □

From these lemmas we have

Theorem 4.1. *Suppose that operators P_η and P_σ commute. If for some fixed point $f \in L^1$ of P_η, the limit*

$$l = \lim_{t\to\infty} S_t f$$

exists, then $\lim_{t\to\infty} T_t f = l$ *and l is a fixed point of P_σ and P_η.*

Proof. Since $P_\eta(f) = f$, using lemma 4.2, we have that

$$\begin{aligned} ||T_t f - l|| &= ||S_t(P_\eta^k f) - l|| \\ &= ||S_t f - l||, \end{aligned}$$

which implies that $\lim_{t\to\infty} T_t f = l$.

From the continuity of P_η we obtain

$$\begin{aligned} P_\eta l &= \lim_{t\to\infty} S_t P_\eta f \\ &= \lim_{t\to\infty} S_t f = l. \end{aligned}$$

To prove that l is fixed point of P_σ, notice that

$$P_\sigma l = P_\sigma(\lim_{t\to\infty} T_t f) = \lim_{t\to\infty} P_\sigma(T_t f).$$

Now, using lemmas 4.4 and 4.5, we obtain

$$P_\sigma l = \lim_{t\to\infty} P_\sigma(T_t f) = \lim_{t\to\infty} T_t(P_\sigma f) = \lim_{t\to\infty} T_t f = l,$$

therefore, $P_\sigma l = l$. $\square$

Remark 4.1. *When P_η has a unique fixed point f, the limit*

$$\lim_{t\to\infty} S_t f$$

exists and is equal to f. Indeed, as P_η and P_σ commute, we have that $P_\sigma(f)$ is a fixed point of P_η, then $P_\sigma(f) = f$ and therefore $S_t f = f$.

Theorem 4.2. *If there exists $f \in L^1$ such that the limit*

$$l = \lim_{t\to\infty} T_t f$$

exists and P_σ commutes with P_η, then the limit l is a fixed point of the operators T_t, P_σ and P_η.

Proof. First, we prove that l is a fixed point of T_t. Let m be a non–negative real number. By lemma 4.3 we have that

$$\|T_t f - l\| \geq \|T_m(T_t f - l)\| = \|T_m T_t f - T_m l\| \geq \|T_{m+t} f - T_m l\|.$$

This inequality implies that $\lim_{t\to\infty} T_{t+m} f = T_m l$. Since $\lim_{t\to\infty} T_{t+m} f = l$ we get

$$T_m l = l.$$

The proof that l is a fixed point of P_σ is the same as the one given in theorem 4.1.

Finally, we prove that l is a fixed point of P_η. From lemma 4.2 we know that $T_t f = S_t(P_\eta^k f)$ for $t \in [k\lambda, (k+1)\lambda)$. Moreover, $S_t l = l$ since $P_\sigma l = l$. Hence,

$$l = \lim_{t\to\infty} T_t l = \lim_{t\to\infty} S_t(P_\eta^k l) = \lim_{t\to\infty} P_\eta^k(S_t l) = \lim_{k\to\infty} P_\eta^k l.$$

From this follows that $P_\eta l = l$. □

With the same argument used in the proof of theorem 4.2, we obtain

Corollary 4.1. *If the operators P_σ and P_η commute and, for some $f \in L^1$, there is a sequence $\{t_n\}_{n=1}^\infty$ such that the limit*

$$f^* = \lim_{n\to\infty} T_{t_n} f$$

exists, then f^* is a fixed point of P_σ and P_η.

When we speak of an absolutely continuous invariant measure ν, this means that ν is absolutely continuous with respect to measure μ.

Proposition 4.2. *Let σ and η be two commutative maps from E into itself. The map σ has an absolutely continuous invariant measure if and only if the map η has an absolutely continuous invariant measure.*

Proof. If ν is an absolutely continuous invariant measure for σ and P_σ is a Frobenius-Perron's operator of σ, then there exists $f \in L^1(E)$ such that $P_\sigma(f) = f$. Since P_η and P_σ commute, it follows that $P_\eta(f)$ is a fixed point of P_σ and the set

$$C = \{g \in L^1(E) \ : \ P_\sigma(g) = g\}$$

is a convex and compact for the weak topology.

By the Markus–Kakutani's theorem (see [DS], page 456) the operators P_σ and P_η have a fixed point in C. Therefore η has an absolutely continuous invariant measure. □

Remark 4.2. *If σ or η has an absolutely continuous invariant measure then it follows from the proof of the above proposition that σ and η have at least one common absolutely continuous invariant measure*

Example 4.1. *If we take $\sigma(x) = 1 - 2x^2$ and $\eta(x) = 4x^3 - 3x$ defined on $E = [-1, 1]$, the corresponding system described in Section 2 has a stationary density.*

In this example σ and η are Chebychev's polynomials, so they commute. In addition, σ admits an absolutely continuous invariant measure ν (see [AR]). By the last remark, the map η has the same absolutely continuous invariant measure ν. In fact, Adler (see [AR]) proved that $\mu = \nu$. By theorem 4.1 the system has a stationary density f and f is the Radon–Nikodym's derivative of μ.

Example 4.2. *The affine maps*

$$\sigma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 1 \\ \frac{1}{2}y + 2 \end{pmatrix} \quad \textit{and} \quad \eta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 2 \\ 2y - 4 \end{pmatrix}$$

are commutative. However, this does not imply that they have a common stationary density.

In this example we see that the transformations σ and η have an unique fixed point $(2, 4)$. This point is a global attractor for σ and global repeller for η. So, σ has a stationary distribution that is the Dirac δ concentrated at the fixed point. It is clear that η does not have a stationary distribution. This does not contradict proposition 4.2 because σ has an invariant measure which is not absolutely continuous.

Moreover we obtain some computational evidence that, for the system corresponding to the interaction of these two linear maps, the existence of a stationary distribution depends on the mean λ of the Poisson distribution. Indeed, for λ greater or equal to 1, the system does not have any stationary distribution since its orbits go to infinity. However, for $\lambda < 1$, the orbits go to the point $(2, 4)$, which is the common fixed point of η and σ. So, in this case the stationary distribution is the Dirac delta concentrated in $(2, 4)$.

This example shows that the hypothesis $f \in L^1$ in theorem 4.1 and 4.2 is necessary.

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